Notes on Nakayama's Lemma, Jacobson Radicals, and Related Topics

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1 Introduction

These notes cover several fundamental notions in ring and module theory, focusing on the *Jacobson radical*, *local rings*, and *Nakayama's lemma*. We also provide analogies to group theory, in particular to the *Frattini subgroup* of a group. Throughout, A will denote a (not necessarily commutative) ring with unity 1_A, and modules are right A-modules unless stated otherwise.

2 Jacobson Radical

Definition 2.1 (Jacobson Radical)**.** Let A be a ring (possibly noncommutative). The *Jacobson radical* of A, denoted $J(A)$, is the intersection of all maximal right ideals of A. Symbolically,

$$J(A) \;=\; \bigcap_{\substack{M \subseteq A \\ M \text{ is a maximal right ideal}}} M.$$

Equivalently (in rings satisfying certain finiteness or one-sided Artinian conditions), it is also the intersection of all maximal left ideals.

If A is *semisimple* (i.e. A is semisimple as a right module over itself), then $J(A) = 0$.

Remark 2.2 (Analogy in Group Theory)**.** There is an analogous notion in group theory: the *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of all maximal subgroups of G. It plays a role similar to the Jacobson radical in ring theory, in the sense that it measures the "largest" subgroup that does not affect generation of G.

2.1 Properties of the Jacobson Radical

Proposition 2.3. *If V is a simple right A-module, then $V \cong A/I$ for some maximal right ideal $I \subseteq A$. In that case,*

$$\operatorname{Ann}(V) \;=\; \operatorname{Ann}(A/I) \;=\; I.$$

Hence every maximal right ideal appears as the annihilator of some simple module.

Proof. This is a standard result in module theory: a simple right A-module V is isomorphic to A/I for some maximal right ideal I. The annihilator of A/I in A is precisely I. □

Corollary 2.4.

$$J(A) \;=\; \bigcap_{\substack{V \\ V \text{ simple}}} \operatorname{Ann}(V).$$

Example 2.5. If $A = M_n(F)$ is the ring of $n \times n$ matrices over a field F, then A is semisimple (by the Artin–Wedderburn theorem). Hence $J(A) = 0$. Indeed, $M_n(F)$ has no nonzero proper two-sided ideals in the semisimple setting.

3 Local Rings

Definition 3.1 (Local Ring)**.** A ring A (not necessarily commutative) is called *local* if it has a unique maximal right ideal (equivalently, a unique maximal left ideal). In that case, this unique maximal right ideal must be $J(A)$ itself. Hence for a local ring A,

$$J(A) \;=\; \text{the unique maximal right ideal of } A.$$

Example 3.2. Any division ring D is local, with $J(D) = \{0\}$. A more instructive example is the ring of p^n-adic integers (commutative local ring) or a power-series ring over a field. In the finite setting, $\mathbb{Z}/p^n\mathbb{Z}$ is a local ring with maximal ideal $p\mathbb{Z}/p^n\mathbb{Z}$.

4 Nakayama's Lemma

Nakayama's lemma is a fundamental result about finitely generated modules over rings whose Jacobson radical has special nilpotency or is contained in a particular ideal. It is sometimes called the "Fitting lemma" in older texts, and is intimately related to the notion of "minimal number of generators" of a module.

4.1 Statement and First Proof Sketch

Theorem 4.1 (Nakayama's Lemma)**.** *Let A be a ring and $I \subseteq J(A)$ be an ideal (two-sided, if A is noncommutative) such that M is a finitely generated right A-module. If $M = IM$, then $M = 0$.*

Equivalently, if N is a submodule of M such that $M = N + IM$, then $M = N$.

Sketch of Proof (Matrix Trick). We sketch a common argument for the case $I = J(A)$. Let M be generated by $m_1, \ldots, m_k$ as a right A-module. Suppose $M = J(A)M$. Then each generator m_i can be written as

$$m_i = \sum_{j=1}^{k} m_j a_{ji}, \quad \text{where } a_{ji} \in J(A).$$

In matrix form, setting $\mathbf{m} = (m_1, \ldots, m_k)^T$ and $A = (a_{ji})_{1 \le j,i \le k}$, we have

$$\mathbf{m} = \mathbf{m} \cdot A, \quad \text{where } A \in M_k(J(A)).$$

Rearrange to get

$$\mathbf{m}(I - A) = 0,$$

where I is the $k \times k$ identity matrix. The key step is to show that $1 - x$ is a unit for all $x \in J(A)$ (under suitable hypotheses, e.g. local or nilpotent conditions). By an appropriate argument (the "Jacobi identity" or a local invertibility trick), one shows $I - A$ is invertible in $M_k(A)$. Hence $\mathbf{m} = 0$, which implies $M = 0$.

The second statement (if $M = N + IM$, then $M = N$) follows by applying the first part to the quotient M/N. □

4.2 An Alternative Proof via Minimal Generators

A more "elementary" proof of Nakayama's lemma in the special case $M = M \cdot J(A)$ (and M is finitely generated) proceeds as follows:

Alternative Proof (Minimal Generators). Let M be a nonzero right A-module which is *finitely generated*, and assume $M = M \cdot J(A)$. Pick a minimal generating set $\{x_1, x_2, \ldots, x_n\}$ for M.

Because $M = M \cdot J(A)$, each generator x_i can be written as

$$x_i = \sum_{k=1}^{n} x_k \, a_{ki}, \quad a_{ki} \in J(A).$$

If we let $X = (x_1, x_2, \ldots, x_n)^T$ and $A = (a_{ki}) \in M_n(J(A))$, the same matrix argument as before applies:

$$X = X \cdot A \quad \Longrightarrow \quad X(I - A) = 0.$$

Under the assumption that $1 - x$ is invertible for $x \in J(A)$ (e.g. A is local or $J(A)$ is nilpotent), we conclude $I - A$ is invertible. Hence $X = 0$, contradicting that $x_1, \ldots, x_n$ were a generating set. Thus M must be zero if $M = M \cdot J(A)$.

This also implies the usual corollary: if $M = N + M \cdot J(A)$ for some submodule $N \subseteq M$, then $M = N$. (Apply the above argument to M/N.) □

4.3 Connection to the Frattini Subgroup

The radical $J(A)$ in ring theory is analogous to the Frattini subgroup $\Phi(G)$ in group theory. Recall:

Definition 4.2 (Frattini Subgroup)**.** For a group G, the *Frattini subgroup* $\Phi(G)$ is the intersection of all maximal subgroups of G. Equivalently, it is the smallest normal subgroup of G such that $G/\Phi(G)$ is generated by any lift of a generating set of $G/\Phi(G)$.

This mirrors the Nakayama lemma's statement that "generators can be reduced modulo the radical." In other words, if a set of elements of M generate M modulo $J(A)M$, then they actually generate M.

4.4 Definition of Semisimple and a Note on $J(A) = 0$

Definition 4.3 (Semisimple Ring)**.** A ring A is called *semisimple* if $J(A) = 0$ and A is Artinian (or equivalently satisfies other finiteness conditions). By the Artin–Wedderburn theorem, a semisimple ring is isomorphic to a finite direct product of matrix rings over division rings.

Remark 4.4. When A is semisimple, Nakayama's lemma becomes trivial for $I \subseteq J(A)$, because $J(A) = 0$. In that case, any relation of the form $M = M \cdot I$ forces $M = 0$ immediately if $I \subseteq \{0\}$.

5 Additional Examples and Remarks

Example 5.1 (A Semisimple Ring)**.** As noted, $M_n(F)$ is semisimple, so $J(M_n(F)) = 0$. Consequently, Nakayama's lemma in this setting is trivial, since the only ideal contained in $J(M_n(F))$ is $\{0\}$.

Example 5.2 (Group Algebras)**.** Let G be a finite group and F a field. The group algebra $F[G]$ is the vector space over F with basis $\{g : g \in G\}$ and multiplication extended linearly from the group multiplication.

- If $\mathrm{char}(F) \nmid |G|$, then by *Maschke's Theorem*, $F[G]$ is semisimple, so $J(F[G]) = 0$.
- If $\mathrm{char}(F)$ divides $|G|$, then $F[G]$ may not be semisimple, and $J(F[G])$ can be nonzero. One often studies the structure of this radical to understand *modular representations* of G.

Example 5.3 (Commutative Local Rings)**.** If A is a commutative local ring with maximal ideal $\mathfrak{m}$ (so $J(A) = \mathfrak{m}$), then Nakayama's lemma often appears in algebraic geometry and commutative algebra, e.g. to prove that a finitely generated module over A whose images of generators in $M/\mathfrak{m}M$ are zero must itself be zero.

Remark 5.4 (Schur's Lemma Connection)**.** In the study of modules over noncommutative rings (particularly group algebras), *Schur's lemma* states that if V is a simple module over a division ring D, then $\mathrm{End}_D(V)$ is itself a division ring. Over $M_n(F)$, all simple modules are isomorphic to F^n (as right modules), and endomorphisms correspond to matrix multiplication, reinforcing that $J(M_n(F)) = 0$.

6 Summary

We have seen:

- The **Jacobson radical** $J(A)$ is the intersection of all maximal right ideals.
- A **local ring** is one with a unique maximal right ideal, which must be $J(A)$.
- **Nakayama's lemma** says that if M is a finitely generated A-module and $I \subseteq J(A)$ is an ideal with $M = IM$, then $M = 0$. Equivalently, if $M = N + IM$ for some submodule $N \subseteq M$, then $M = N$.
- The radical $J(A)$ is analogous to the **Frattini subgroup** in group theory, which is the intersection of all maximal subgroups.
- Over **semisimple** rings (like $M_n(F)$ when F is a field), the radical vanishes ($J(A) = 0$).

These concepts form part of the foundation of modern ring and module theory, with deep connections to representation theory (especially of finite groups) and to commutative algebra in the local case.

References

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